On Euler's "Misleading Induction", Andrews' "Fix", and How to Fully Automate them

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Dedicated to George Andrews on his $(75-\epsilon)^{th}$ *birthday*

Recall that the *trinomial coefficient*([Wei])

$$\binom{n}{j}_2$$

is the coefficient of x^j in

$$(1+x+x^{-1})^n \quad .$$

In other words,

$$(1+x+x^{-1})^n = \sum_{j=-n}^{n} \binom{n}{j}_2 x^j \quad .$$

Also recall that the *Fibonacci numbers*, F_n, are defined by $F_{-1}=1, F_0=0$, and $F_n = F_{n-1}+F_{n-2}$ for $n>0$.

The fascinating story of how Euler *almost* fooled himself into believing that

$$3\binom{n+1}{0}_2 - \binom{n+2}{0}_2 = F_n(F_n+1) \quad , \qquad (Leonhard)$$

for **all** n because he checked this for the **nine** values $-1 \le n \le 7$, only to find out that it fails for $n=8$, leading him to record it for *posterity* in [E], has been told several times, including the nice 'popular' book by David Wells[Wel], Eric Weisstein's extremely useful Mathworld website[Wei], and Ed Sandifer's famous on-line MAA column[S].

In the first three sections of George Andrew's important article [An] (that merely serve as the motivation and background for the remaining sections that talk about deep q-analogs), he describes a brilliant way to 'correct' the left side of of $(Leonhard)$ in order to make the identity come true for *all* $n \ge -1$.

First he used the obvious fact that

$$\binom{n+2}{0}_2 = \binom{n+1}{-1}_2 + \binom{n+1}{0}_2 + \binom{n+1}{1}_2$$

(and symmetry) to rewrite Eq. $(Leonhard)$ as:

$$\binom{n+1}{0}_2 - \binom{n+1}{1}_2 = \frac{1}{2}F_n(F_n+1) \quad , \qquad (Leonhard')$$

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and then went on to prove (using ad-hoc human ways) the identity

$$\sum_{j=-\infty}^{\infty} \binom{n+1}{10j}_2 - \sum_{j=-\infty}^{\infty} \binom{n+1}{10j+1}_2 = \frac{1}{2} F_n(F_n+1) \quad . \qquad (George)$$

Note that for $n < 8$ the only non-zero summands in $(George)$ are with $j = 0$.

[At the risk of giving away the punch-line, let's remark that once conjectured, a fully rigorous proof of $(George)$ can be obtained by checking it, à la Euler, for (to be safe) $0 \leq n \leq 20$.]

Interlude: Even Giants make stupid conjectures

We are a little surprised that Euler could have believed, *even for a second*, that Eq. $(Leonhard)$ is true for *all* n. Completely by hand (see [S]) Euler found a three-term linear recurrence with *polynomial* coefficients for $\binom{n}{0}_2$ (that easily implies such a recurrence for $3\binom{n+1}{0}_2 - \binom{n+2}{0}_2$, both are easily found today with the *Almkvist-Zeilberger algorithm* [AlZ]), so he should have realized that it can't equal $F_n(F_n+1)$ that satisfies a linear recurrence with *constant* coefficients (in other words it is what is called today a C-finite sequence, see [Z]).

Another way Euler could have easily realized that $(Leonhard)$ is false is via asymptotics, even a very crude one. The ratio of consecutive terms on the left side of $(Leonhard)$ obviously tends to 3 while the ratios of consecutive terms on the right side tends to $\phi^2 = 2.61803\ldots$.

The General case

With Maple (or Sage, or any computer algebra system), it is a piece of cake to generate many Euler-style cautionary tales, and Andrews-style fixes. Let's summarize our findings by stating a general theorem, whose **proof** also tells you an **algorithm** how to compute rational generating functions for these sequences. This algorithm has been implemented in the Maple package `GEA` available from

`http://www.math.rutgers.edu/~zeilberg/tokhniot/GEA` .

Theorem: Let $P(x)$ be any Laurent polynomial, and let

$$\binom{n}{j}_P$$

be the coefficient of x^j in $P(x)^n$.

Let k be a positive integer, and let a be an integer such that $0 \leq a < k$. Define the *generalized Euler-Andrews* sum to be

$$A(n,k,a) := \sum_{j=-\infty}^{\infty} \binom{n}{kj+a}_P \quad .$$

The *generating functions*

$$f_{k,a}(t) := \sum_{n=0}^{\infty} A(n,k,a) t^n \quad , \quad (0 \leq a < k) \quad ,$$

are **rational functions** of t, all with the **same** denominator, of degree k in t. They are easily computable by linear algebra.

Equivalently, the $k-1$ sequence $\{A(n,k,a)\}_{n=0}^{\infty}$ $(0 \leq a \leq k-1)$ satisfy the **same** homogeneous linear recurrence equation with **constant** coefficients of order k (but of course with (usually) different initial conditions).

Proof: Let's spell-out $P(x)$

$$P(x) = \sum_{i=\alpha}^{\beta} c_i x^i \quad ,$$

where $\alpha < \beta$ (and α may be negative, of course). Then, obviously, we have the analog of Pascal's triangle identity:

$$\binom{n}{j}_P = \sum_{i=\alpha}^{\beta} c_i \binom{n-1}{j-i}_P \quad .$$

Hence

$$A(n,k,a) = \sum_{i=\alpha}^{\beta} c_i A(n-1,k,a-i \ \ mod \ \ k) \quad .$$

On the level of generating functions we get

$$f_{k,a}(t) := \delta_{a,0} + t \sum_{i=\alpha}^{\beta} c_i f_{k,(a-i) \ mod \ k} \quad , \quad (0 \leq a < k) \quad ,$$

where $\delta_{a,0}$ is 1 when $a=0$ and 0 otherwise. This gives us a system of k linear equations in the k unknowns

$$\{f_{k,0}(t), f_{k,1}(t), \ldots f_{k,k-1}(t)\} \quad ,$$

that Maple can solve very fast. The fact that the (same) denominator has degree k, (and the numerators have degree $k-1$) follows from Cramer's rule.

This is implemented in procedure `GA(P,x,k,t)`, in the Maple package `GEA`, that inputs a Laurent polynomial P in the variable x, a positive integer k and a variable t, and outputs a list of rational functions in t, of length k, whose $(a+1)$-th entry is $f_{k,a}(t)$. If P is symmetric ($P(x) = P(1/x)$) one only has to go as far as $a \leq k/2$, since then $f_{k,k-a}(t) = f_{k,a}(t)$. This (faster) case is handled by the procedure `GAs(P,x,k,t)`.

Computerized Redux of Andrews's man-made proof

The case $P(x) = x^{-1} + 1 + x$ and $k = 10$ is the one that Andrews needed. So all you need is type

`GAs(x+1+1/x,x,10,t);` ,

giving all the six generating functions, whose coefficients, $A(m,10,a)$ $(0 \leq a \leq 5)$, are given explicitly in Eq. (2.18) of [An] (reproduced in [Wei]) as expressions involving Fibonacci numbers and powers of 3. It follows from the theorem (even **without** actually computing the generating

functions!) that Andrews' claimed formulas can be **proved rigorously** by 'just' checking the first 20 special cases, as remarked above.

Ditto for Theorem 2.1 of [An], (also quoted in [Wei]). An empirical proof à la Euler (and now one can manage with $m \leq 10$) suffices.

But if you do not know beforehand conjectured expressions, then procedure `GA` can find the generating functions *ab initio*.

The Beauty of Programming

Even Euler and Andrews would soon get tired of doing the analogous thing for other k. Andrews also did the case $k = 6$ in Theorem 3.1 of [An], but we can do it for *all* k up to 100 (easily) and not just for $P(x) = x^{-1} + 1 + x$, but for *any* $P(x)$. See the sample output in the front of this article

`http://www.math.rutgers.edu/~zeilberg/mamarim/mamarimhtml/gea.html` .

In particular, we can generate (**many!**) Euler-Style *cautionary tales* about the premature use of empirical induction, see procedure `BCT` and the webbook

`http://www.math.rutgers.edu/~zeilberg/tokhniot/oGEA6` .

Encore: Probabilistic Interpretation

Suppose you have a (fair or) loaded die whose faces are marked with dollar amounts (some positive, some negative, some (possibly) 0), at each throw you 'gain' the amount on the landed face (of course if the amount is 0 you get nothing, and if the amount is negative, you have to pay). Let $a(n)$ be the probability of breaking even after n throws. Then of course the generating function of $a(n)$ is **not** rational, i.e. the sequence $a(n)$ does **not** satisfy a linear recurrence with **constant** coefficients (on the other hand it does satisfy a linear recurrence with **polynomial** coefficients, easily found by the Almkvist-Zeilberger[AlZ] algorithm, implemented in the Maple package `http://www.math.rutgers.edu/~zeilberg/tokhniot/EKHAD.`).

But fixing k (even a very large one, say a googol), then the related probability, let's call it $b_k(n)$ of getting an exact multiple of k (that for small n is the same as breaking even), does satisfy a linear recurrence equation with **constant** coefficients of order k, (equivalently, the generating function is a rational function of degree k). Ditto for the probability $b_{k,a}(n)$ of finishing with an amount that leaves remainder a when divided by k.

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